به نام خدا

سیگنال‌ها و سیستم‌ها

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Lecture-9

SIGNALS
&
SYSTEMS

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Laplace Transform

- Based on Lecture slides by

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MIT OpenCourseWare
6.003 Signals and Systems

Desirable Characteristics of a Set of "Basic" Signals

a. We can represent large and useful classes of signals using these building blocks

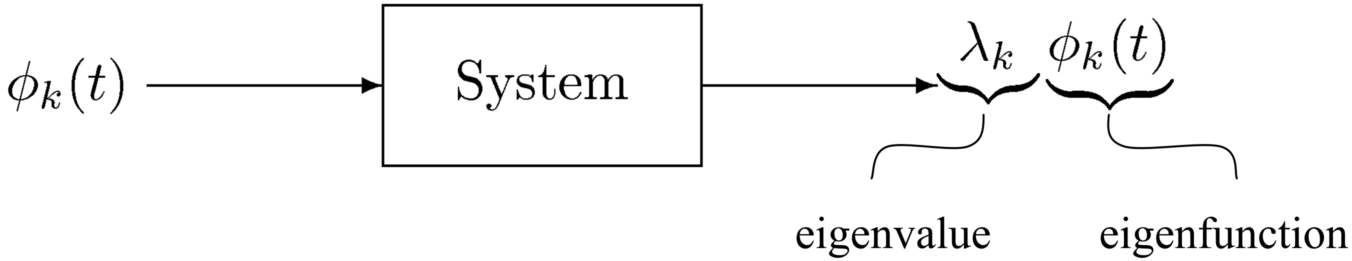
b. The response of LTI systems to these basic signals is particularly simple, useful, and insightful

Previous focus: Unit samples and impulses

Focus now: Eigenfunctions of all LTI systems

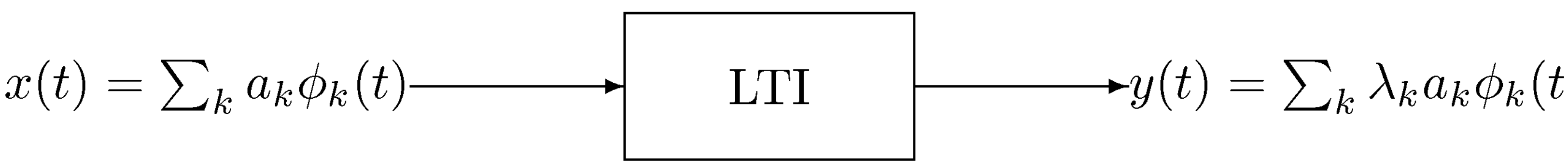
The eigenfunctions $\phi_k(t)$ and their properties

(Focus on CT systems now, but results apply to DT systems as well.)



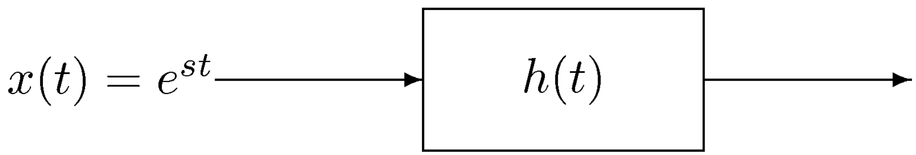
Eigenfunction in $\rightarrow$ same function out with a "gain"

From the superposition property of LTI systems:



Now the task of finding response of LTI systems is to determine λ_k.

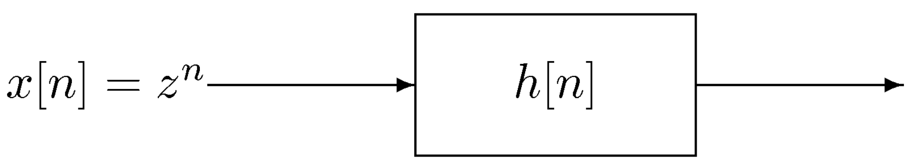
Complex Exponentials as the Eigenfunctions of any LTI Systems



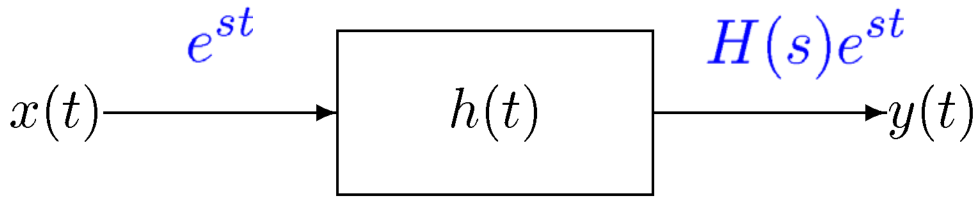
$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = \left[\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \right] e^{st} = \underbrace{H(s)}_{\text{eigenvalue}} \underbrace{e^{st}}_{\text{eigenfunction}}$$

— e^{st} is an eigenfunction of *any* LTI system

— $s = \sigma + j\omega$ can be complex in general



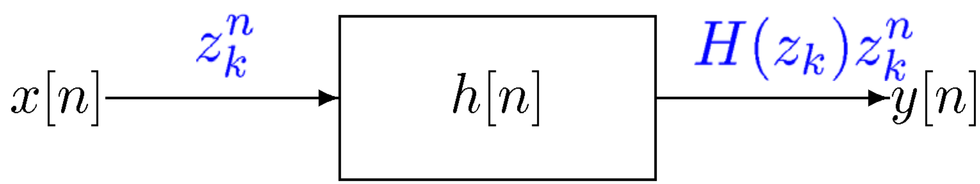
$$y[n] = \sum_{m=-\infty}^{\infty} h[m] z^{n-m} = \left[\sum_{m=-\infty}^{\infty} h[m] z^{-m} \right] z^n = \underbrace{H(z)}_{\text{eigenvalue}} \underbrace{z^n}_{\text{eigenfunction}}$$



$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

$$x(t) = \sum_k a_k e^{s_k t} \longrightarrow y(t) = \sum_k H(s_k) a_k e^{s_k t}$$

DT:



$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$x[n] = \sum_k a_k z_k^n \longrightarrow y[n] = \sum_k H(z_k) a_k z_k^n$$

The (Bilateral) Laplace Transform

$$x(t) \longleftrightarrow X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \mathcal{L}\{x(t)\}$$

$s = \sigma + j\omega$ is a *complex* variable – Now we explore the full range of s

Basic ideas:

absolute integrability needed

(1) $$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

(2) A critical issue in dealing with Laplace transform is convergence:
— $X(s)$ generally exists only for *some* values of s, located in what is called the *region of convergence* (ROC)

$$\text{ROC} = \{s = \sigma + j\omega \text{ so that } \int_{-\infty}^{\infty} \underbrace{|x(t)e^{-\sigma t}|}_{\substack{\text{Depends}\\ \text{only on } \sigma\\ \text{not on } \omega}} dt < \infty\}$$

absolute integrability condition

(3) If $s = j\omega$ is in the ROC (i.e. $\sigma = 0$), then

$$X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\}$$

Properties of the Laplace Transform

Property	**Signal**	**Transform**	**ROC**
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R [i.e., s is in the ROC if $(s - s_0)$ is in R]
Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$	"Scaled" ROC (i.e., s is in the ROC if (s/a) is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\} > 0\}$

Initial- and Final Value Theorems

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

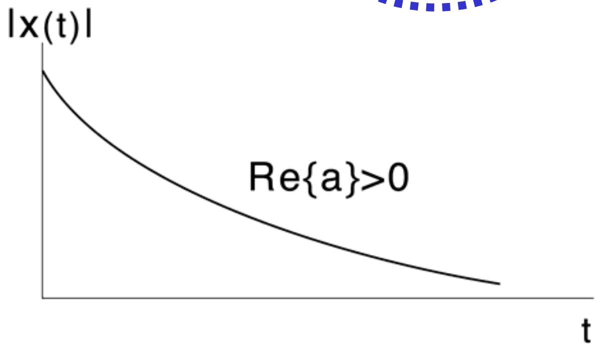
$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

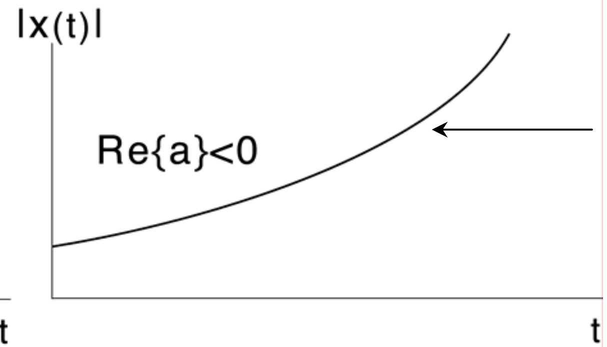
Laplace Transforms of Elementary Functions

Signal	Transform	ROC
1. $\delta(t)$	1	All s
2. $u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3. $-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4. $\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5. $-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6. $e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\alpha$
7. $-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\alpha$
8. $\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\alpha$
9. $-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\alpha$
10. $\delta(t-T)$	e^{-sT}	All s
11. $[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12. $[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13. $[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
14. $[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\alpha$
15. $u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16. $u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

Example #1:

$$x_1(t) = e^{-at}u(t) \qquad (a - \text{an arbitrary real or complex number})$$





Unstable:
- no *Fourier Transform*
- but *Laplace Transform* exists

$$X_1(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_0^{\infty} e^{-(s+a)t}dt$$

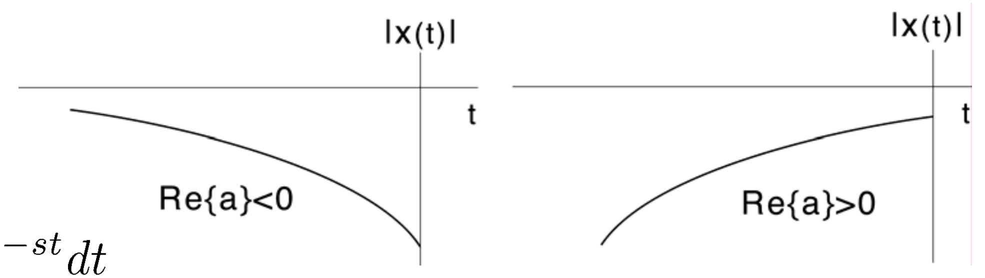
$$= -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = -\frac{1}{s+a}[e^{-(s+a)\infty} - 1]$$

This converges only if $Re(s+a) > 0$, i.e. $Re(s) > -Re(a)$

$$\Downarrow$$

$$X_1(s) = \frac{1}{s+a}, \quad \underbrace{\Re e\{s\} > -\Re e\{a\}}_{\text{ROC}}$$

Example #2:



$$x_2(t) = -e^{-at}u(-t)$$

$$X_2(s) = -\int_{-\infty}^{\infty} e^{-at}u(-t)e^{-st}dt$$

$$= -\int_{-\infty}^{0} e^{-(s+a)t}dt$$

$$= +\frac{1}{s+a}e^{-(s+a)t}\Big|_{-\infty}^{0} = \frac{1}{s+a}[1-e^{(s+a)\infty}]$$

This converges only if *Re*(*s*+*a*) < 0, i.e. *Re*(*s*) < -*Re*(*a*)

$$X_2(s) = \frac{1}{s+a}, \quad \underbrace{\Re e\{s\} < -\Re e\{a\}}_{\text{ROC}}$$ - Same as $X_1(s)$, but different ROC

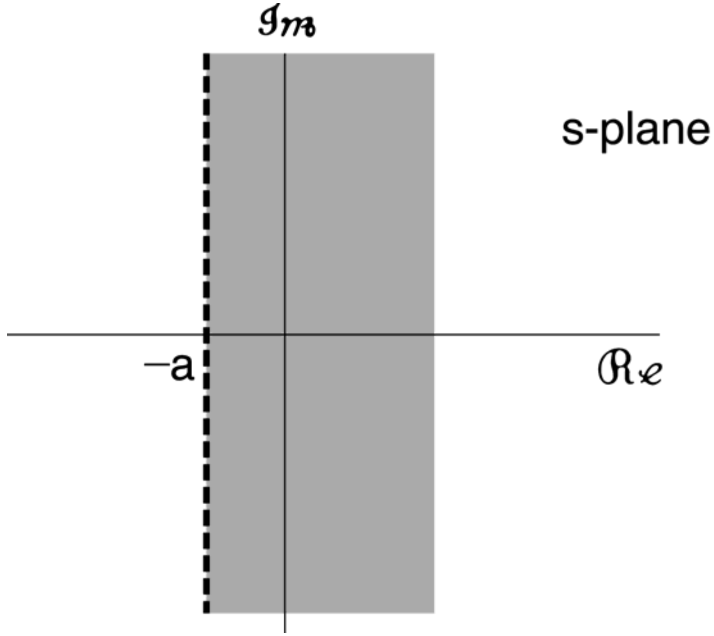
Key Point (and key difference from *FT*): Need *both X*(*s*) *and* ROC to uniquely determine *x*(*t*). No such an issue for *FT*.

Graphical Visualization of the ROC

Example #1

$$X_1(s) = \frac{1}{s+a}, \quad \Re e\{s\} > -\Re e\{a\}$$

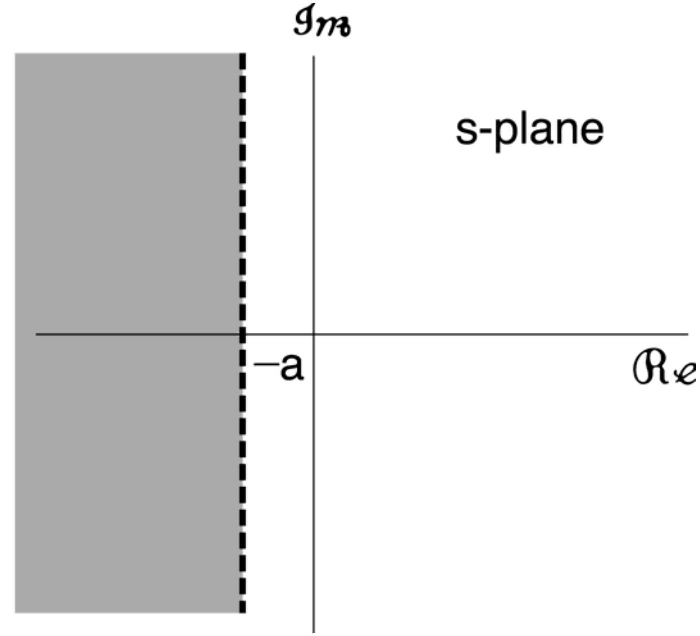
$x_1(t) = e^{-at}u(t)$ - right-sided signal



Example #2

$$X_2(s) = \frac{1}{s+a}, \quad \Re e\{s\} < -\Re e\{a\}$$

$x_2(t) = -e^{-at}u(-t)$ - left-sided signal



Rational Transforms

- Many (but by no means all) Laplace transforms of interest to us are rational functions of s (e.g., Examples #1 and #2; in general, impulse responses of LTI systems described by LCCDEs), where

$$X(s) = \frac{N(s)}{D(s)}, \quad N(s), D(s) - \text{polynomials in } s$$

- Roots of $N(s)$ = *zeros* of $X(s)$

- Roots of $D(s)$ = *poles* of $X(s)$

- Any $x(t)$ consisting of a linear combination of complex exponentials for $t > 0$ and for $t < 0$ (e.g., as in Example #1 and #2) has a rational Laplace transform.

Example #3 $x(t) = 3e^{2t}u(t) - 2e^{-t}u(t)$

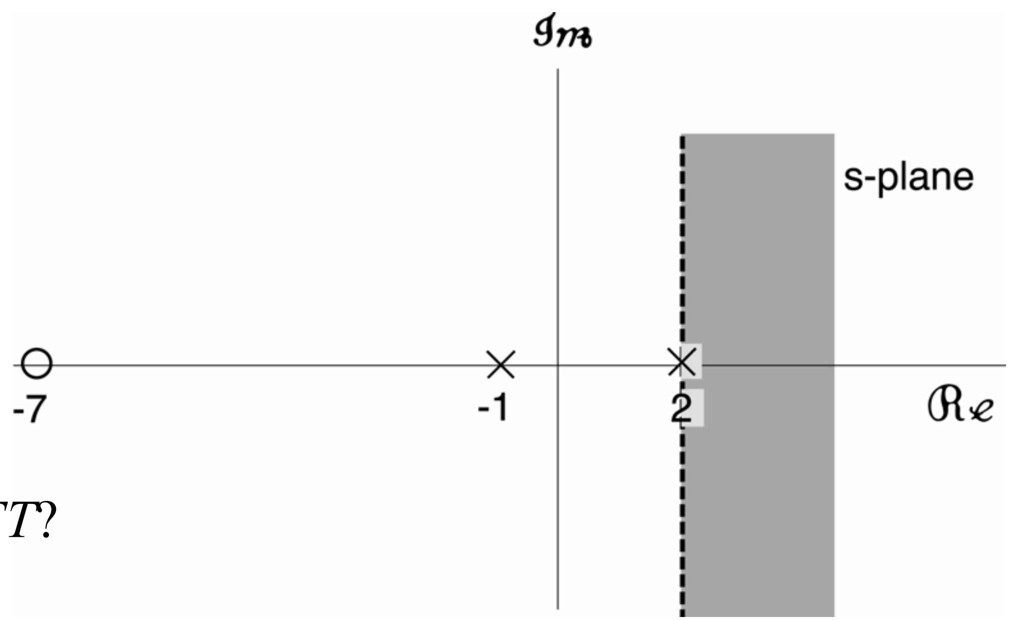
$x_1(t) = e^{-at}u(t)$ - right-sided signal $\quad X_1(s) = \dfrac{1}{s+a}, \quad \Re e\{s\} > -\Re e\{a\}$

$$X(s) = \frac{3}{s-2} - \frac{2}{s+1} = \frac{s+7}{(s-2)(s+1)} = \frac{s+7}{s^2-s-2} \quad \Re e\{s\} > 2$$

Notation:

× — *pole*

○ — *zero*



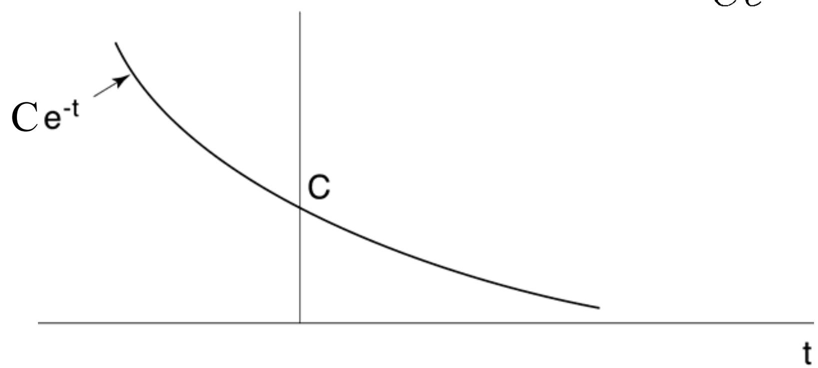
Q: Does $x(t)$ have FT?

Laplace Transforms and ROCs

- Some signals do not have Laplace Transforms (have no ROC)

(a) $x(t) = Ce^{-t}$ for all t since $\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt = \infty$ for all σ

$$|x(t)e^{-\sigma t}| = Ce^{-(\sigma+1)t}$$



(b) $x(t) = e^{j\omega_0 t}$ for all t $\quad$ *FT*: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt = \int_{-\infty}^{\infty} e^{-\sigma t} dt = \infty \text{ for all } \sigma$$

$X(s)$ is defined only in ROC; we don't allow impulses in LTs

Properties of the ROC

- The ROC can take on only a small number of different forms

1) The ROC consists of a collection of lines parallel to the $j\omega$-axis in the s-plane (i.e. the ROC only depends on σ).
   Why?

$$\int_{-\infty}^{\infty} |x(t)e^{-st}|dt = \int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|dt < \infty \text{ depends only on } \sigma = \Re e\{s\}$$

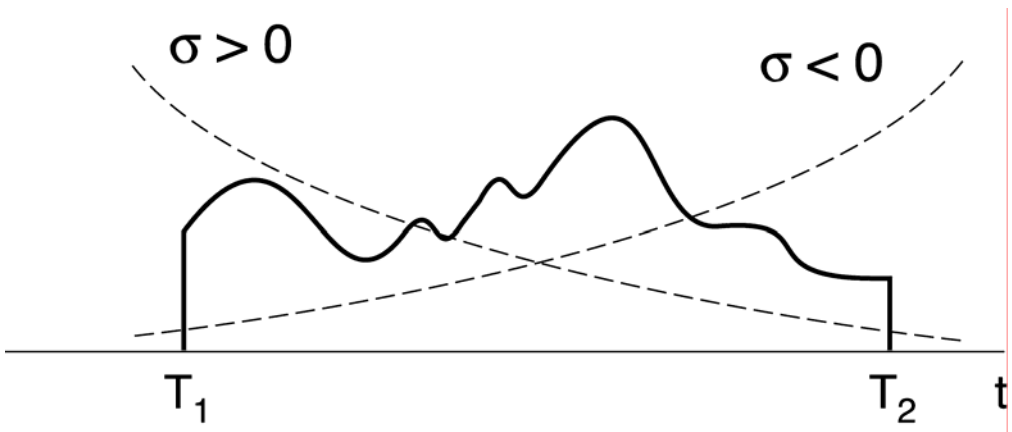
2) If $X(s)$ is rational, then the ROC does not contain any poles.
   Why?

$$\text{Poles are places where } D(s) = 0$$

$$\Rightarrow \ X(s) = \frac{N(s)}{D(s)} = \infty \quad \text{Not convergent.}$$

More Properties

3) If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

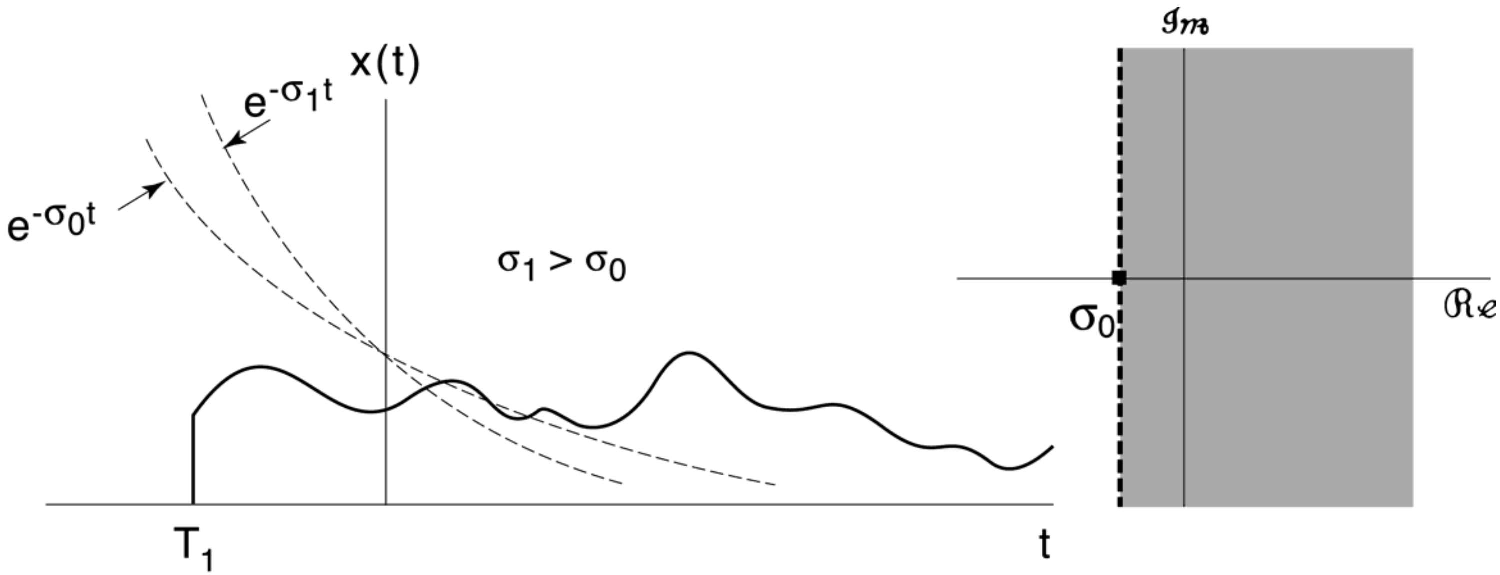


$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \underbrace{\int_{T_1}^{T_2} x(t)e^{-st}dt}_{\text{A finite integration interval}}$$

$$< \infty \quad \text{if} \int_{T_1}^{T_2} |x(t)|dt < \infty$$

ROC Properties that Depend on Which Side You Are On - I

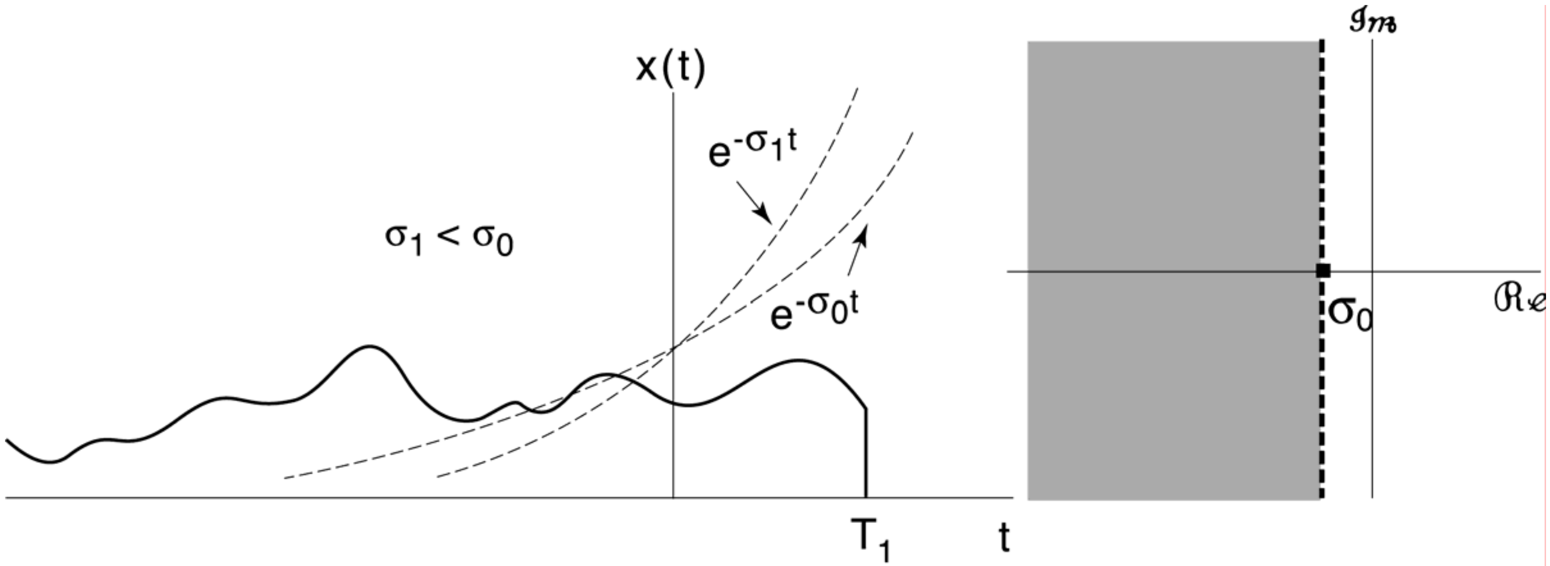
4) If $x(t)$ is right-sided (i.e. if it is zero *before* some time), and if $Re(s) = \sigma_o$ is in the ROC, then all values of s for which $Re(s) > \sigma_o$ are also in the ROC.



ROC is a right half plane (RHP)

ROC Properties that Depend on Which Side You Are On - II

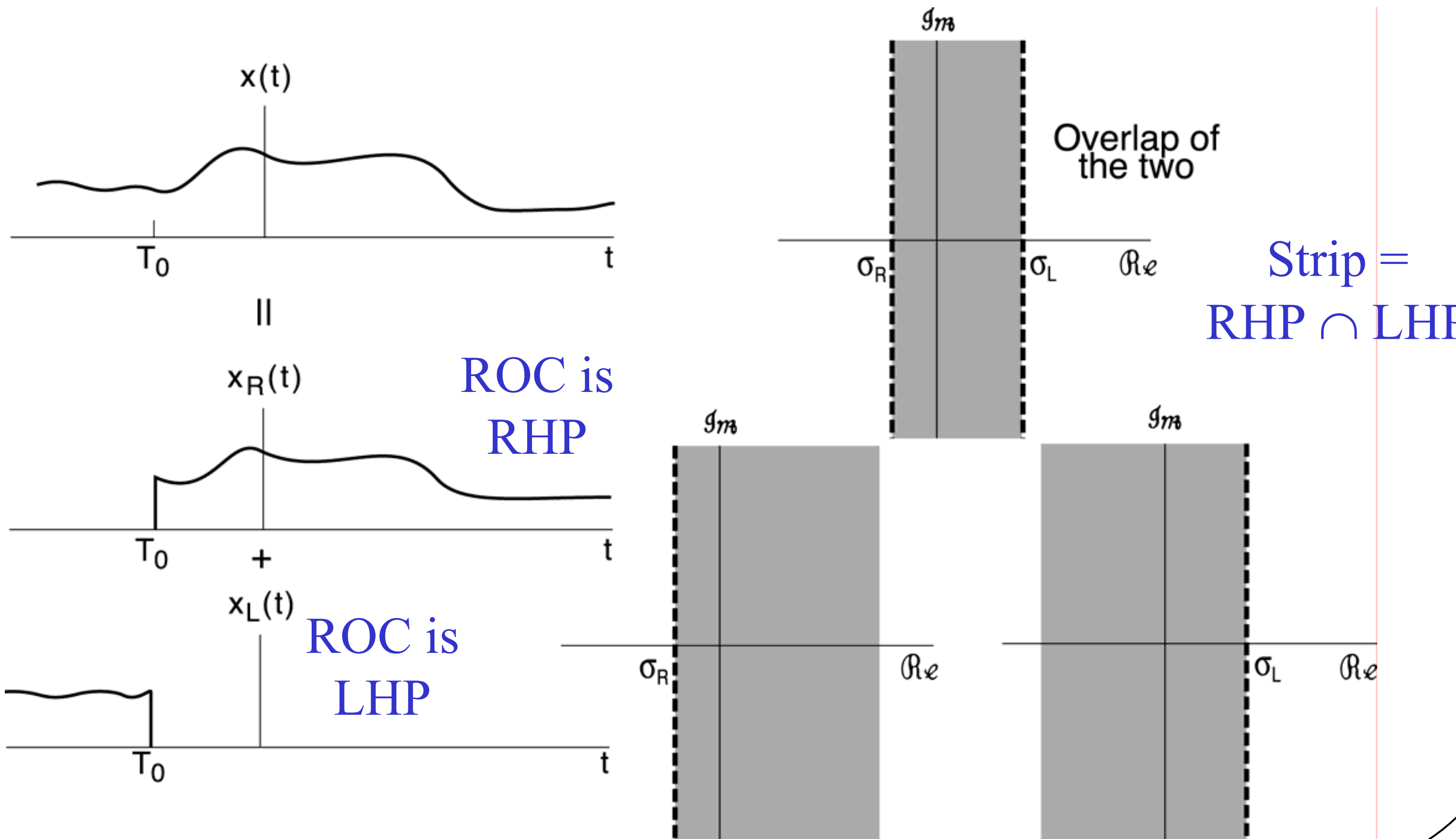
5) If $x(t)$ is left-sided (i.e. if it is zero *after* some time), and if $Re(s) = \sigma_o$ is in the ROC, then all values of s for which $Re(s) < \sigma_o$ are also in the ROC.



ROC is a left half plane (LHP)

Still More ROC Properties

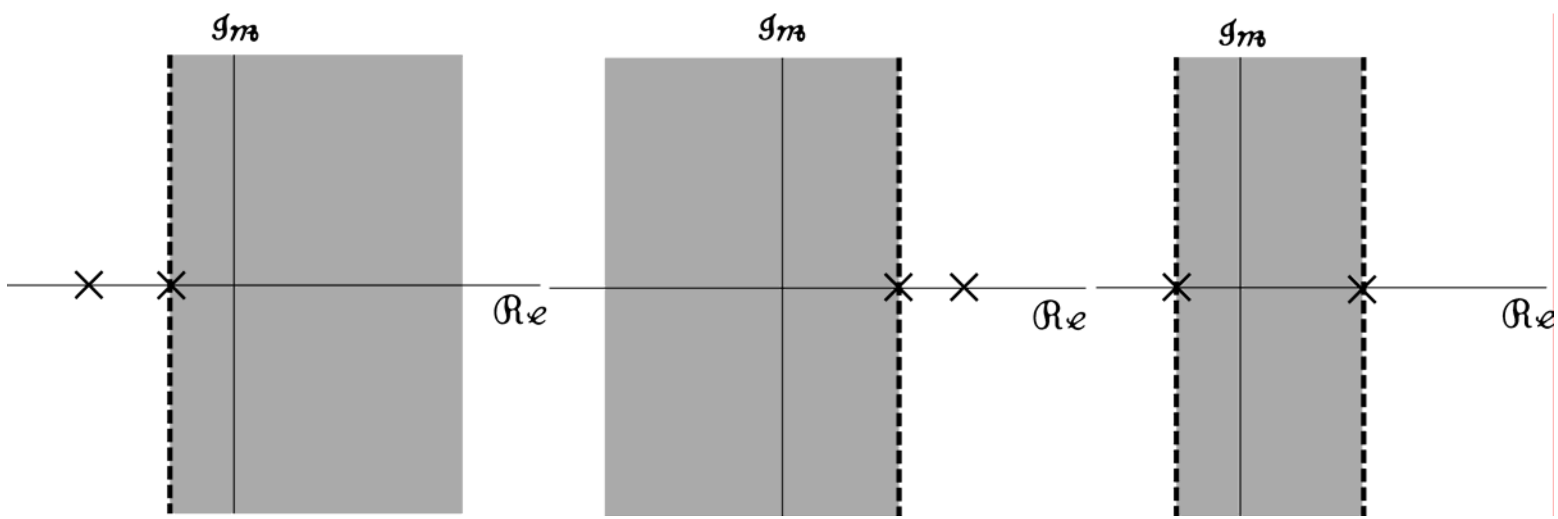
6) If $x(t)$ is two-sided and if the line $Re(s) = \sigma_o$ is in the ROC, then the ROC consists of a strip in the s-plane that includes the line $Re(s) = \sigma_o$.



Properties, Properties

7) If $X(s)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

8) Suppose $X(s)$ is rational, then
   (a) If $x(t)$ is right-sided, the ROC is to the right of the rightmost pole.
   (b) If $x(t)$ is left-sided, the ROC is to the left of the leftmost pole.



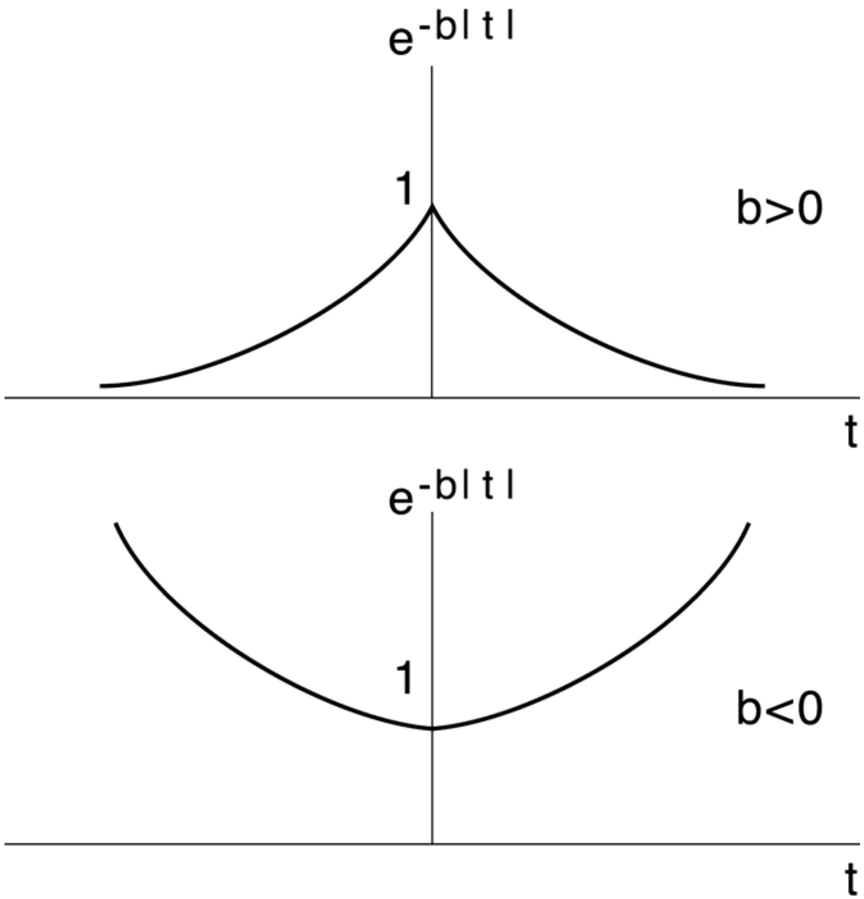
(a) (b) (c)

9) If ROC of $X(s)$ includes the $j\omega$-axis, then *FT* of $x(t)$ exists.

Example:

$$x(t) = e^{-b|t|}$$

Intuition?



- Okay: multiply by constant (e^{0t}) and will be integrable

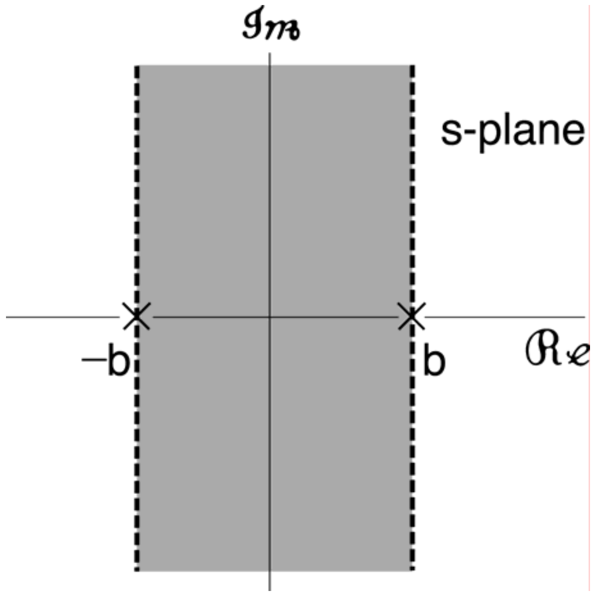
- Looks bad: no $e^{\sigma t}$ will dampen both sides

Example (continued):

$$x(t) = e^{bt}u(-t) + e^{-bt}u(t)$$

$$e^{bt}u(-t) \Downarrow -\frac{1}{s-b},\ \Re e\{s\} < b \qquad e^{-bt}u(t) \Downarrow \frac{1}{s+b},\ \Re e\{s\} > -b$$

Overlap if $b > 0 \Rightarrow X(s) = \frac{-2b}{s^2-b^2}$, with ROC:



What if $b < 0$? $\Rightarrow$ No overlap $\Rightarrow$ No Laplace Transform

Inverse Laplace Transform

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt, \quad s = \sigma + j\omega \in \text{ ROC} \\ &= \mathcal{F}\{x(t)e^{-\sigma t}\} \end{aligned}$$

Fix $\sigma \in$ ROC and apply the inverse Fourier transform

$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

$$\Downarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$$

But $s = \sigma + j\omega$ (σ fixed) $\Rightarrow ds = jd\omega$

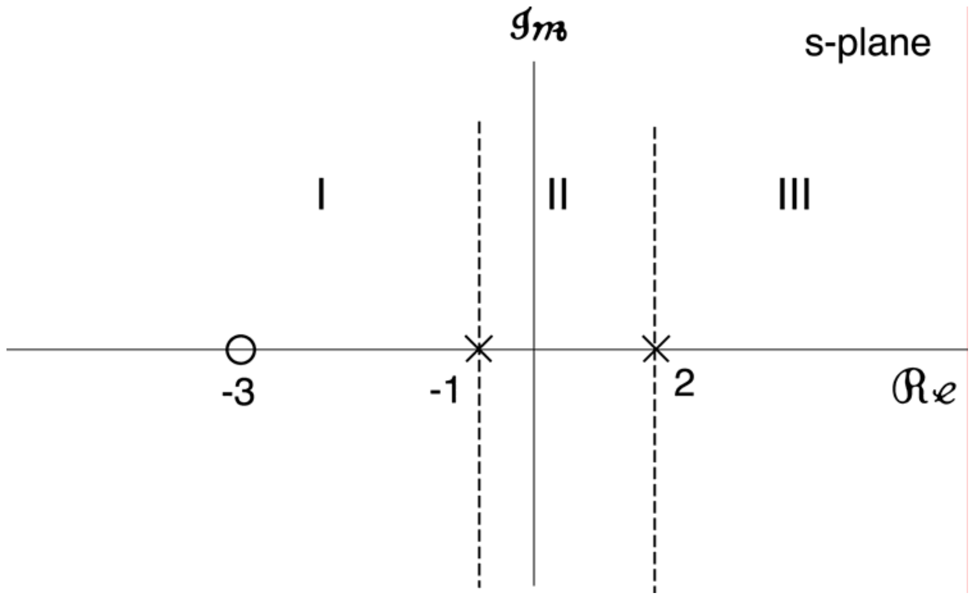
$$\Downarrow$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} X(s)e^{st}ds$$

Example:

$$X(s) = \frac{(s+3)}{(s+1)(s-2)}$$

Three possible ROCs:



			Fourier Transform exists?
$x(t)$ is right-sided	ROC:	III	No
$x(t)$ is left-sided	ROC:	I	No
$x(t)$ extends for all time	ROC:	II	Yes

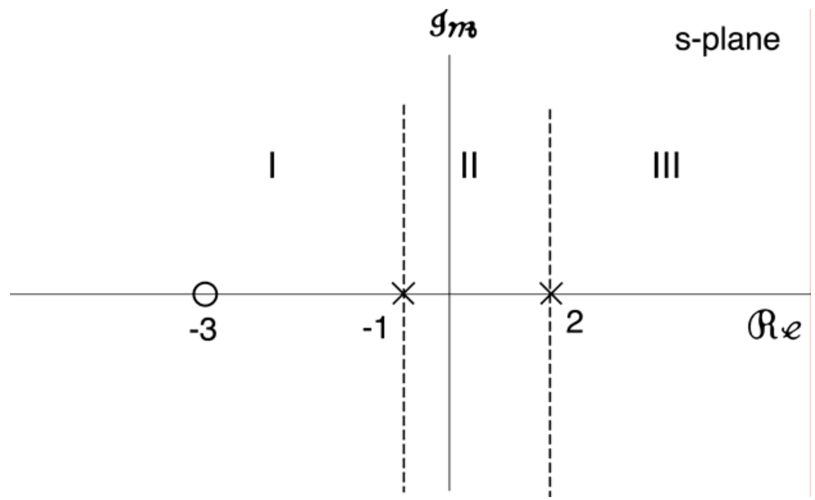
Inverse Laplace Transforms Via Partial Fraction Expansion and Properties

Example:

$$X(s) = \frac{s+3}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

$$A = -\frac{2}{3}, \quad B = \frac{5}{3}$$

Three possible ROC's — corresponding to three *different* signals



Recall $\quad \dfrac{1}{s+a}, \quad \Re e\{s\} < -a \longleftrightarrow -e^{-at}u(-t)$ left-sided

$\dfrac{1}{s+a}, \quad \Re e\{s\} > -a \longleftrightarrow e^{-at}u(t)$ right-sided

ROC I: — Left-sided signal.

$$x(t) = -Ae^{-t}u(-t) - Be^{2t}u(-t)$$
$$= \left[\frac{2}{3}e^{-t} - \frac{5}{3}e^{2t}\right]u(-t) \qquad \text{Diverges as } t \to -\infty$$

ROC II: — Two-sided signal, has Fourier Transform.

$$x(t) = Ae^{-t}u(t) - Be^{2t}u(-t)$$
$$= -\left[\frac{2}{3}e^{-t}u(t) + \frac{5}{3}e^{2t}u(-t)\right] \qquad \to 0 \text{ as } t \to \pm\infty$$

ROC III:— Right-sided signal.

$$x(t) = Ae^{-t}u(t) + Be^{2t}u(t)$$
$$= \left[-\frac{2}{3}e^{-t} + \frac{5}{3}e^{2t}\right]u(t) \qquad \text{Diverges as } t \to +\infty$$

Properties of Laplace Transforms

Linearity

$$ax_1(t) + bx_2(t) \longleftrightarrow aX_1(s) + bX_2(s)$$

ROC at least the intersection of ROCs of $X_1(s)$ and $X_2(s)$

* ROC can be bigger (due to pole-zero cancellation)

E.g. $x_1(t) = x_2(t)$ and $a = -b$

Then $ax_1(t) + bx_2(t) = 0 \longrightarrow X(s) = 0$

$\Rightarrow$ ROC entire s-plane

Time Shift

$$x(t-T) \longleftrightarrow e^{-sT}X(s), \text{ same ROC as } X(s)$$

Example:

$$\frac{e^{3s}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad ?$$

$$\frac{e^{-sT}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad e^{-2t}u(t)|_{t\to t-T}$$

$$\downarrow T = -3$$

$$\frac{e^{3s}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad e^{-2(t+3)}u(t+3)$$

Time-Domain Differentiation

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st}ds, \quad \frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} sX(s)e^{st}ds$$

$$\Downarrow$$

$$\frac{dx(t)}{dt} \longleftrightarrow sX(s), \text{ with ROC containing the ROC of } X(s)$$

ROC could be bigger than the ROC of $X(s)$, if there is pole-zero cancellation. E.g.,

$$\begin{array}{ccccll} x(t) & = & u(t) & \leftrightarrow & \frac{1}{s}, & \Re e\{s\} > 0 \\ \frac{dx(t)}{dt} & = & \delta(t) & \leftrightarrow & 1 = s \cdot \frac{1}{s} & \text{ROC} = \text{entire s-plane} \end{array}$$

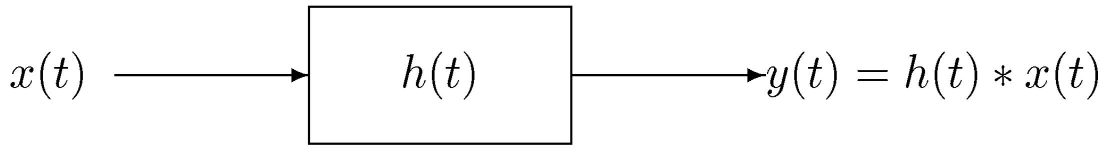
s-Domain Differentiation

$$-tx(t) \leftrightarrow \frac{dX(s)}{ds}, \text{ with same ROC as } X(s)$$

(Derivation is similar to $\frac{d}{dt} \leftrightarrow s$)

E.g., $te^{-at}u(t) \leftrightarrow -\frac{d}{ds}\left[\frac{1}{s+a}\right] = \frac{1}{(s+a)^2}, \quad \Re e\{s\} > -a$

Convolution Property



For $x(t) \longleftrightarrow X(s), y(t) \longleftrightarrow Y(s), h(t) \longleftrightarrow H(s)$

Then $$Y(s) = H(s) \cdot X(s)$$

- ROC of $Y(s) = H(s)X(s)$: at least the overlap of the ROCs of $H(s)$ & $X(s)$
- ROC could be empty if there is no overlap between the two ROCs
  E.g.

$$x(t) = e^{t}u(t), \text{ and } \quad h(t) = -e^{-t}u(-t)$$

- ROC could be larger than the overlap of the two. E.g.

$$x(t) * h(t) = \delta(t)$$

Initial- and Final-Value Theorems

If $x(t) = 0$ for $t < 0$ and there are no impulses or higher order discontinuities at the origin, then

$$x(0^+) \quad = \quad \lim_{s\to\infty} sX(s)$$

Initial value

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

$$x(\infty) \quad = \quad \lim_{s\to 0} sX(s)$$

Final value

Applications of the Initial- and Final-Value Theorem

For
$$X(s) = \frac{N(s)}{D(s)}$$

n - order of polynomial $N(s)$, d - order of polynomial $D(s)$

- Initial value:

$$x(0^+) = \lim_{s\to\infty} sX(s) = \begin{cases} 0 & d > n+1 \\ \text{finite } \neq 0 & d = n+1 \\ \infty & d < n+1 \end{cases}$$

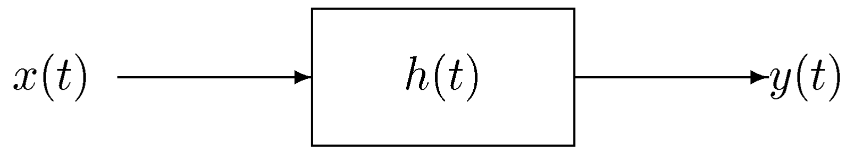
E.g. $X(s) = \dfrac{1}{s+1}$ $\quad x(0^+) = ?$

- Final value

$$\text{If } x(\infty) = \lim_{s\to 0} sX(s) = 0 \Rightarrow \lim_{s\to 0} X(s) < \infty$$

$$\Rightarrow \text{No poles at } s = 0$$

The System Function of an LTI System



$h(t) \quad \longleftrightarrow \quad H(s)$ – the system function

The system function characterizes the system

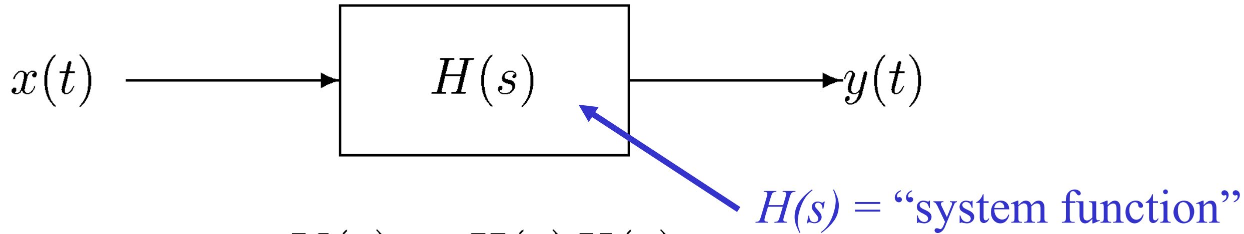
$\Downarrow$

System properties correspond to properties of $H(s)$ and its ROC

A first example:

$$\text{System is stable} \Leftrightarrow \int_{-\infty}^{\infty} |h(t)|dt < \infty \quad \Leftrightarrow \quad \begin{array}{l}\text{ROC of } H(s) \\ \text{includes the } j\omega \text{ axis}\end{array}$$

CT System Function Properties



$$Y(s) = H(s)X(s)$$

1) System is stable $\Leftrightarrow \int_{-\infty}^{\infty} |h(t)|dt < \infty \Leftrightarrow$ ROC of $H(s)$ includes $j\omega$ axis

2) Causality $\Rightarrow$ $h(t)$ right-sided signal $\Rightarrow$ ROC of $H(s)$ is a right-half plane

Question:

If the ROC of $H(s)$ is a right-half plane, is the system causal?

Ex.

$$H(s) = \frac{e^{sT}}{s+1}, \quad \Re e\{s\} > -1 \Rightarrow h(t) \text{ right-sided}$$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{e^{sT}}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}_{t \to t+T} = e^{-t}u(t)|_{t \to t+T}$$

$$= e^{-(t+T)}u(t+T) \neq 0 \quad \text{at} \quad t < 0$$

Non-causal

Properties of CT Rational System Functions

a) However, if $H(s)$ is ***rational,*** then

The system is causal $\Leftrightarrow$ The ROC of $H(s)$ is to the right of the rightmost pole

b) If $H(s)$ is rational and is the system function of a causal system, then

The system is stable $\Leftrightarrow$ $j\omega$-axis is in ROC $\Leftrightarrow$ all poles are in LHP

Checking if All Poles Are In the Left-Half Plane

$$H(s) = \frac{N(s)}{D(s)}$$

Poles are the roots of $D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$

Method #1: Calculate all the roots and see!
Method #2: Routh-Hurwitz – Without having to solve for roots.

	Polynomial	Condition so that all roots are in the LHP
First-order	$s + a_0$	$a_0 > 0$
Second-order	$s^2 + a_1 s + a_0$	$a_1 > 0, a_0 > 0$
Third-order	$s^3 + a_2 s^2 + a_1 s + a_0$	$a_2 > 0, a_1 > 0, a_0 > 0$ <u>and</u> $a_0 < a_1 a_2$
	$\vdots$	$\vdots$

LTI Systems Described by LCCDEs

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Repeated use of differentiation property: $\frac{d}{dt} \leftrightarrow s, \quad \frac{d^k}{dt^k} \leftrightarrow s^k$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$\Downarrow$$

$$Y(s) = H(s)X(s)$$

where $H(s) = \underbrace{\frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}}_{\text{Rational}}$

$\longleftarrow$ roots of numerator $\Rightarrow$ *zeros*

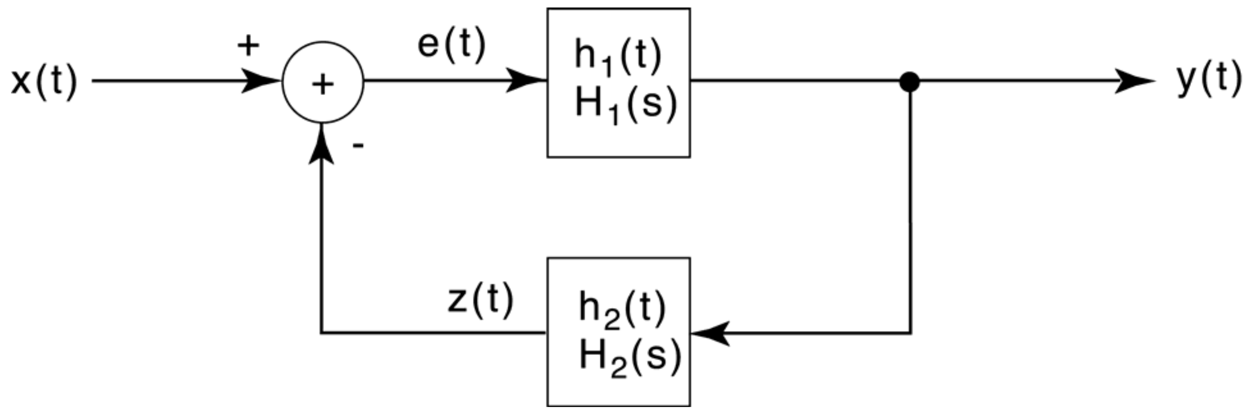
$\longleftarrow$ roots of denominator $\Rightarrow$ *poles*

ROC =? Depends on:

1) Locations of *all* poles.
2) Boundary conditions, *i.e.* right-, left-, two-sided signals.

System Function Algebra

Example: A basic feedback system consisting of ***causal*** blocks



$$E(s) = X(s) - Z(s) = X(s) - H_2(s)Y(s)$$

$$Y(s) = H_1(s)E(s) = H_1(s)[X(s) - H_2(s)Y(s)]$$

$$\Downarrow$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

← More on this later in feedback

ROC: Determined by the roots of $1+H_1(s)H_2(s)$, instead of $H_1(s)$

Block Diagram for Causal LTI Systems with Rational System Functions

Example:

$$Y(s) = H(s)X(s)$$

$$H(s) = \frac{2s^2+4s-6}{s^2+3s+2} = \left(\frac{1}{s^2+3s+2}\right)(2s^2+4s-6)$$

— Can be viewed as cascade of two systems.

Define:

$$W(s) = \frac{1}{s^2+3s+2}X(s)$$

$$\frac{d^2w(t)}{dt^2} + 3\frac{dw(t)}{dt} + 2w(t) = x(t), \quad \text{initially at rest}$$

$$\text{or} \quad \frac{d^2w(t)}{dt^2} = x(t) - 3\frac{dw(t)}{dt} - 2w(t)$$

Similarly

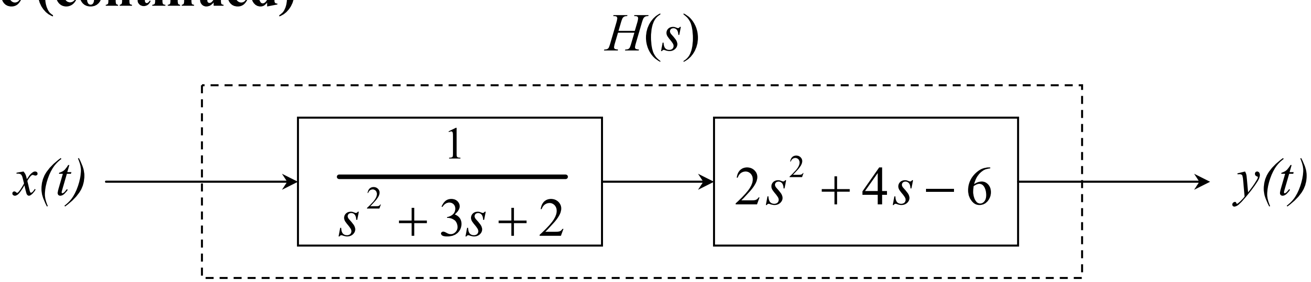
$$Y(s) = (2s^2+4s-6)W(s)$$

$$\Downarrow$$

$$y(t) = 2\frac{d^2w(t)}{dt^2} + 4\frac{dw(t)}{dt} - 6w(t)$$

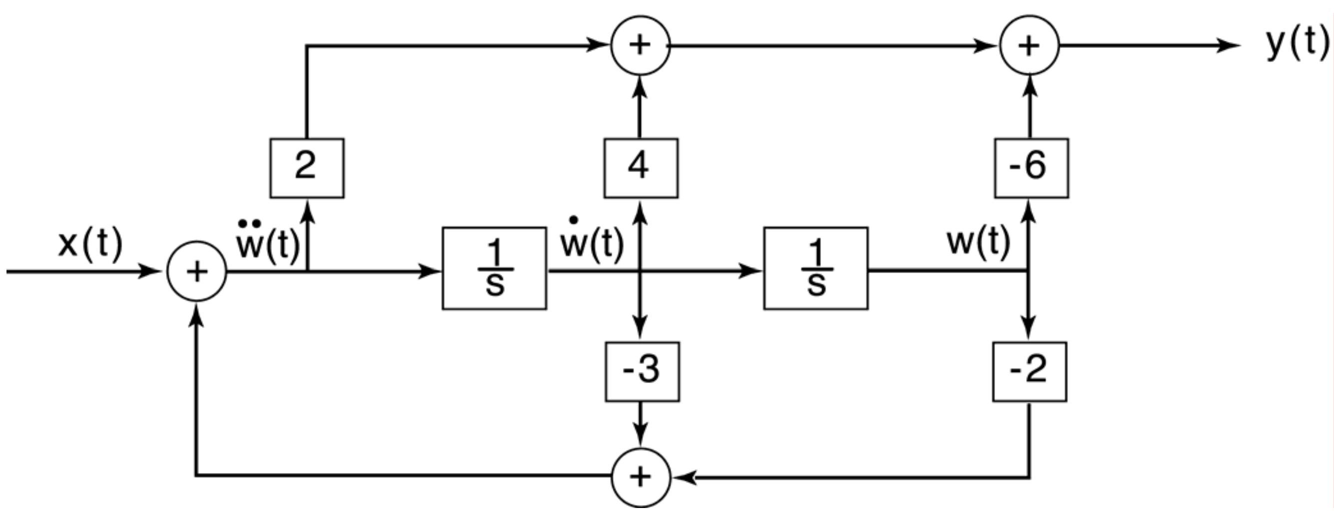
Example (continued)

Instead of



We can construct $H(s)$ using:

$$
\begin{aligned}
\frac{d^2 w(t)}{dt^2} &= x(t) - 3\frac{dw(t)}{dt} - 2w(t) \\
y(t) &= 2\frac{d^2 w(t)}{dt^2} + 4\frac{dw(t)}{dt} - 6w(t)
\end{aligned}
$$

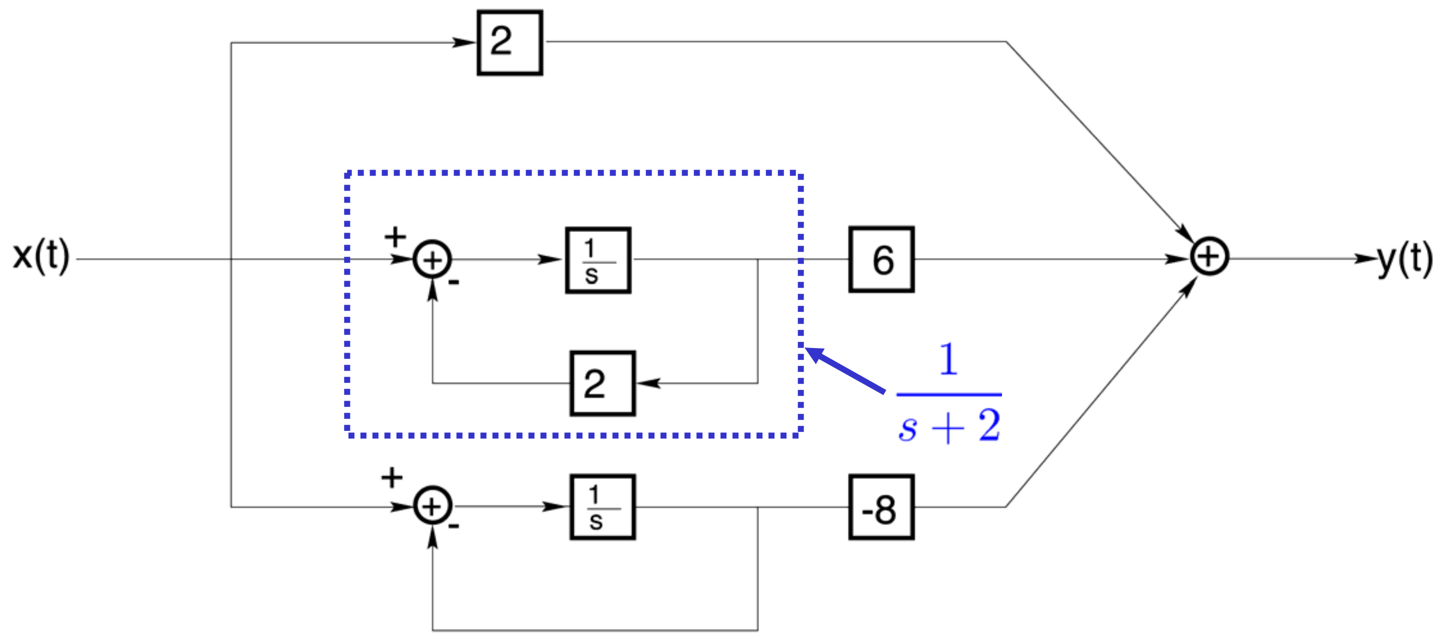


Notation: $1/s$ — an integrator

Note also that

$$H(s) = \left[\frac{2(s-1)}{s+2}\right]\left[\frac{s+3}{s+1}\right] = \left[\frac{s+3}{s+2}\right]\left[\frac{2(s-1)}{s+1}\right] \quad - \text{ Cascade}$$

$$\overset{PFE}{=} \quad 2 + \frac{6}{s+2} - \frac{8}{s+1} \quad - \text{ parallel connection}$$



Lesson to be learned: There are many *different* ways to construct a system that performs a certain function.

The Unilateral Laplace Transform

(The preferred tool to analyze causal CT systems described by LCCDEs with **initial conditions**)

$$\mathcal{X}(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt = \mathcal{UL}\{x(t)\}$$

Note:

1) If $x(t) = 0$ for $t < 0$, then $X(s) = \mathcal{X}(s)$

2) Unilateral LT of $x(t)$ = Bilateral LT of $x(t)u(t\text{-})$

3) For example, if $h(t)$ is the impulse response of a causal LTI system, then

$$H(s) = \mathcal{H}(s)$$

4) Convolution property: If $x_1(t) = x_2(t) = 0$ for $t < 0$, then

$$\boxed{\mathcal{UL}\{x_1(t) * x_2(t)\} = \mathcal{X}_1(x)\mathcal{X}_2(s)}$$

Same as Bilateral Laplace transform

Differentiation Property for Unilateral Laplace Transform

$$x(t) \longleftrightarrow \mathcal{X}(s)$$

$$\Downarrow$$

$$\frac{dx(t)}{dt} \longleftrightarrow s\mathcal{X}(s) - x(0^-)$$

Initial condition!

Derivation:

integration by parts

$\int f \cdot dg = fg - \int g \cdot df$

$$\mathcal{UL}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt \;=\; s\underbrace{\int_{0^-}^{\infty} x(t) e^{-st} dt}_{\mathcal{X}(s)} \;+\; x(t)e^{-st}\Big|_{0-}^{\infty}$$

$$= \; s\mathcal{X}(s) - x(0^-)$$

Note:

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\left\{\frac{dx(t)}{dt}\right\} \;\longleftrightarrow\; s\overbrace{(s\mathcal{X}(s) - x(0^-))}^{\mathcal{UL}\frac{dx(t)}{dt}} - x'(0^-)$$

$$\longleftrightarrow\; s^2\mathcal{X}(s) - sx(0^-) - x'(0^-)$$

Use of ULTs to Solve Differentiation Equations with Initial Conditions

Example:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

$$y(0^-) = \beta\,,\; y'(0^-) = \gamma\,,\; x(t) = \alpha u(t)$$

Take ULT:

$$\underbrace{s^2\mathcal{Y}(s) - \beta s - \gamma}_{\mathcal{UL}\left\{\frac{d^2y}{dt^2}\right\}} + 3\underbrace{(\mathcal{Y}(s) - \beta)}_{\mathcal{UL}\left\{\frac{dy}{dt}\right\}} + 2\mathcal{Y}(s) = \frac{\alpha}{s}$$

$$\Downarrow$$

$$\mathcal{Y}(s) = \underbrace{\frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)}}_{ZIR} + \underbrace{\frac{\alpha}{s(s+1)(s+2)}}_{ZSR}$$

ZIR — Response for zero input $x(t)=0$

ZSR — Response for zero state, $\beta=\gamma=0$, initially at rest

Example (continued)

- Response for LTI system initially at rest ($\beta = \gamma = 0$)

$$\Downarrow$$

$$\mathcal{H}(s) = \frac{\mathcal{Y}(s)}{\mathcal{X}(s)} = \frac{1}{(s+1)(s+2)} = H(s)$$

- Response to initial conditions alone ($\alpha = 0$).
  For example:

$$x(t) = 0 \,(\text{no input}), \quad y(0^-) = 1, \quad y'(0^-) = 0 \quad (\beta = 1, \gamma = 0)$$

$$\Downarrow$$

$$\mathcal{Y}(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$$

$$\Downarrow$$

$$y(t) = 2e^{-t} - e^{-2t}, \quad t \geq 0$$